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EXAMPLES IN THE THEORY OF FUNCTIONS.*

BY MAXIME BÔCHER.

1. $f_1(z)$ and $f_2(z)$ are analytic functions. Let $F(z) = f_1(z) \cdot f_2(z)$. What can we say about $F(z)$ at a point a if $f_1(z)$ and $f_2(z)$ both have poles at a? If $f_1(z)$ and $f_2(z)$ both have essentially singular points at a?

2. If a polynomial in x and y with real coefficients satisfies Laplace's equation, prove that it is the real part of a polynomial in $x + yi$.

Prove that the necessary and sufficient condition that a homogeneous polynomial of the n^{th} degree in x and y satisfy Laplace's equation, is that the equation formed by setting the polynomial equal to zero represent n real straight lines making angles π/n with one another.

3.† A function $u(x, y,)$ is harmonic‡ in a certain region of the plane. Prove that it can be developed about any point (x_0, y_0) of this region in a series of the form:

$$u(x, y) = \Phi_0 + \Phi_1 + \Phi_2 + \cdots$$

where each term is itself harmonic and where Φ_n is a homogeneous integral rational function of the n^{th} degree of $(x - x_0)$ and $(y - y_0)$.

State this proposition using polar coordinates (r, ϕ), the point (x_0, y_0)

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* In teaching any branch of mathematics it is essential not merely to illustrate the theory by examples, but to give the student exercises in considerable number to work out for himself. Moreover, these exercises should not all be capable of solution by *direct* application of the particular principle last considered; the student should learn from them to do something more than merely " follow copy." It is often difficult to find a sufficient variety of such examples, especially in the more recently developed portions of mathematics, and it requires no little labor on the part of the instructor to invent them.

It is with the hope of assisting teachers who experience this difficulty that the following collection of problems has been prepared. If it prove useful we shall hope to publish, at irregular intervals, similar collections in other branches of mathematics; and we shall welcome contributions of single problems or sets of problems for this purpose. We have not intentionally included examples which are contained in any of the ordinary text-books on the subject, but aside from this no restriction has been laid on their source, some being taken with more or less modification from various books and memoirs, while others are entirely original.

[ED.]

† It is suggested that the propositions in 3 and 4 be established by deducing them from similar theorems concerning analytic functions of a complex variable.

‡ A function is said to be harmonic in a region if it is single-valued and continuous throughout this region, has continuous first and second partial derivatives at every point of the region, and at every point of the region satisfies Laplace's equation.

being taken as origin. Determine the coefficients in the series thus obtained in the form of line integrals.

4.* How must the theorem of the last problem be modified if (x_0, y_0) is an isolated singular point *i. e.* if $u(x,y)$ is harmonic everywhere in the neighborhood of (x_0, y_0), but not at (x_0, y_0) itself?

Hence distinguish between the various kinds of isolated singular points of harmonic functions, and discuss the behavior of the function in the neighborhood of each kind of singular point.

In particular prove that if u becomes infinite for every method of approaching (x_0, y_0), it is of the form $C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + u_1(x,y)$ where u_1 is harmonic at (x_0, y_0) as well as elsewhere.

5. Prove geometrically† that if we have two circles C_1 and C_2 in the complex plane, and invert the plane first with regard to C_1 and then with regard to C_2, the total effect will be the same as that of a certain linear transformation $w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.

Prove that reflexion of the complex sphere in the plane of any great circle C corresponds to inversion of the complex plane with regard to the stereographic projection of C.

Prove that rotation (through any angle) of the complex sphere about a diameter, corresponds to a certain linear transformation.

(Suggestion: show that the rotation is equivalent to the succession of two reflexions in diametral planes.)

6. The linear transformation $z' = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ will in general leave two points of the plane fixed. Find the necessary and sufficient condition that infinitesimal portions of the plane surrounding those points should be:

(α) turned but not stretched,
(β) stretched but not turned,

by the transformation.

Obtain from the above a necessary condition that the linear transformation should represent a rotation of the sphere.

7. Prove that if it is possible to define sin z and cos z as analytic func-

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* See foot-note marked † on previous page.

† For an analytical proof see *Forsyth*, p. 523.

tions of z which reduce when z is real to the ordinary sine and cosine, the following formulae will hold:

$$\sin^2 z + \cos^2 z = 1,$$
$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \sin(z_1 + z_2).$$

Avoid the use of any explicit formulae for $\sin z$ and $\cos z$ in this proof.

8. Given two points a and b in the complex plane, a curve C connecting them, and a function $f(t)$ continuous for all values of t which lie on this curve. Prove that

$$F(z) = \int_a^b \frac{f(t)}{t-z}\,dt$$

(the integral being taken along C) is a monogenic function of z at all points of the plane which do not lie on C. What kind of a point does $F(z)$ have at infinity?

9. The function $f(z)$ is analytic throughout a region S; a is a point of this region; and b is any point of the plane. Prove that in the neighborhood of a $f(z)$ can be developed in the form $\sum_{n=0}^{n=\infty} C_n\left(\frac{z-a}{z-b}\right)^n$. What can we say about the region throughout which this series converges and represents $f(z)$?

10. Given the functions $u_0(z)$, $u_1(z)$, $u_2(z)$, . . . analytic throughout a certain finite region S of the z-plane. Given further that the series:

$$u_0(z) + u_1(z) + u_2(z) + \cdots$$

converges uniformly throughout S. Denote the function defined by this series by $f(z)$, and let a be a root of the k^{th} order of $f(z)$. Prove that if a circle be drawn about a as centre and of radius so small that it lies wholly within S and does not include or pass through any root of $f(z)$ except a, then the function represented by the sum of the first n terms of the above series will have exactly k roots in this circle for sufficiently large values of n.

11. In the ring-shaped region bounded by two circles with centres at the origin and radii r_1 and r_2 the function $f(z)$ is single-valued and analytic except for a finite number of poles. Moreover this function joins on continuously to continuous boundary values at all points of both circles, and these boundary values are the same at the two points where any radius cuts the two circles. Prove that this function admits of analytic extension all over the plane.

Determine the function completely in case there are no poles in the original region; and in other cases obtain such properties of the function as you can.

12. $f(z)$ is analytic on a piece of a four-sheeted Riemann's Surface bounded by a large circle and having two branch points of the first order at a_1 connecting the first and second and the third and fourth sheets respectively, and a branch point at a_2 connecting the second and third sheets. Construct a piece of a Riemann's Surface of the same size and shape upon which $\sqrt[3]{(z-a_1)(z-a_2)}\cdot f(z)$ is analytic.

13. Construct the Riemann's Surface for the following functions:

$$(a) \qquad w=\sqrt{\sin z},$$

$$(b) \qquad w=\sqrt{\tan z},$$

$$(c) \qquad w=\sqrt{e^z}.$$

Describe the singularities of each of these functions.

14. Study by the method of conformal representation the function:

$$w=\int_0^z (1-z^2)^{-\frac{2}{3}}dz$$

and its inverse.

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